The Hang Seng University of Hong Kong

Department of Mathematics, Statistics and Insurance

Online Bridging Course 2021 – 2022

Reference Materials

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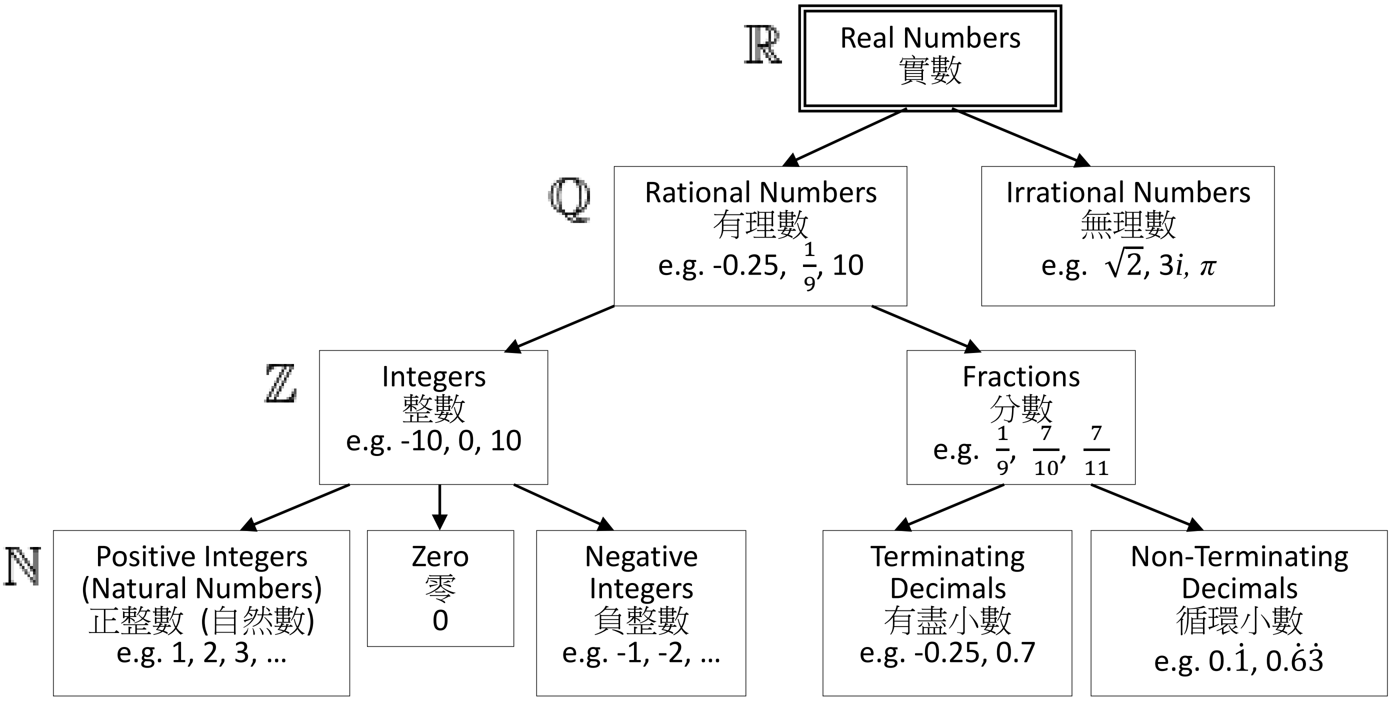
Topic 1 – Numbers and Mathematical Notations

1.1 Real number system



1.2 Mathematical Notations

Symbol	**How it is read**	**Sample Expression**
$\mathbb{R}^2$	2-dimensinal coordinate system	xy-plane
$\in$	... is an element of a set ...	$x \in \mathbb{R}$
$\subset$	... is a proper subset of ...	$\mathbb{N} \subset \mathbb{Q}$
$\{\ \ \}$	... the set ...	Even number = {2, 4, 6, 8, …}
$\mid$	... such that ...	Negative number = $\{x \in \mathbb{R} \mid x < 0\}$
$\lvert\ \rvert$	… absolute value (modulus) …	$\lvert -3 \rvert = 3$
!	… factorial …	n! = n(n-1)…(3)(2)(1)
$\forall$	For all …	$\forall x \mid x < 0$ or $x > -1$
$\exists$	There exists …	$\exists x \mid x > 4$ and $x < 5$
[a, b]	... the closed interval ...	[a, b] mean $\{ x \in \mathbb{R} \mid a \le x \le b\}$
(a, b)	... the open interval ...	(a, b) mean $\{ x \in \mathbb{R} \mid a < x < b\}$
f : A→B	function f from domain A codomain B	f: $\mathbb{Z} \to \mathbb{N} \cup \{0\}$ be defined by $f(x) = x^2$
$\sum_{i=1}^{n} x_i$	sum over ... from ... to ... of	$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$

Practice:

1. Find the member in the set $\left\{-5, -\frac{5}{3}, 0, \sqrt{5},\ \pi, \frac{50}{7}, \sqrt{625}\right\}$ which belongs to $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ respectively.
2. Evaluate a) $|0.01 - 2|$ b) $8! - 3!$
3. State the reciprocal of a) 8 b) $\frac{2}{5} + \frac{1}{2}$
4. Write out explicitly a) $\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{f_i}$ b) $\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{f_i x_i}$
5. By writing out the terms explicitly, show that $\sum_{\mathrm{k}=1}^{5} 3\mathrm{k} = 3 \sum_{\mathrm{k}=1}^{5} \mathrm{k}$

Topic 2 - Indices, Exponential and Logarithmic Functions

2.1 Indices and Radicals

Exponent Properties

$$a^n a^m = a^{n+m} \qquad \frac{a^n}{a^m} = a^{n-m} = \frac{1}{a^{m-n}}$$

$$\left(a^n\right)^m = a^{nm} \qquad a^0 = 1,\quad a \neq 0$$

$$(ab)^n = a^n b^n \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$a^{-n} = \frac{1}{a^n} \qquad \frac{1}{a^{-n}} = a^n$$

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n} \qquad a^{\frac{n}{m}} = \left(a^{\frac{1}{m}}\right)^n = \left(a^n\right)^{\frac{1}{m}}$$

Properties of Radicals

$$\sqrt[n]{a} = a^{\frac{1}{n}} \qquad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[nm]{a} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[n]{a^n} = a, \text{ if } n \text{ is odd}$$

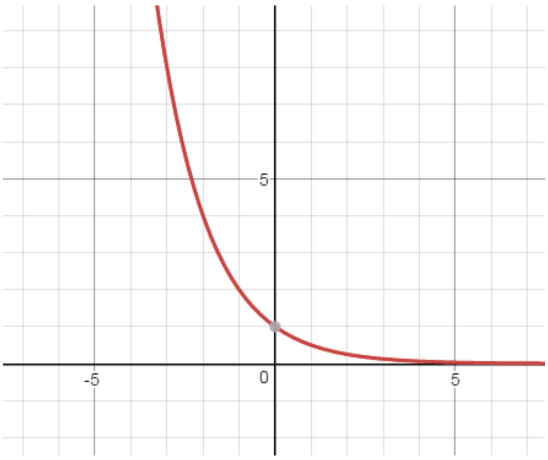
$$\sqrt[n]{a^n} = |a|, \text{ if } n \text{ is even}$$

2.2 Exponential function

An exponential function is $f(x) = a^x$ where a is any value greater than 0, and x is called the exponent. When $a = 1$, the graph is a horizontal line at $y = 1$.

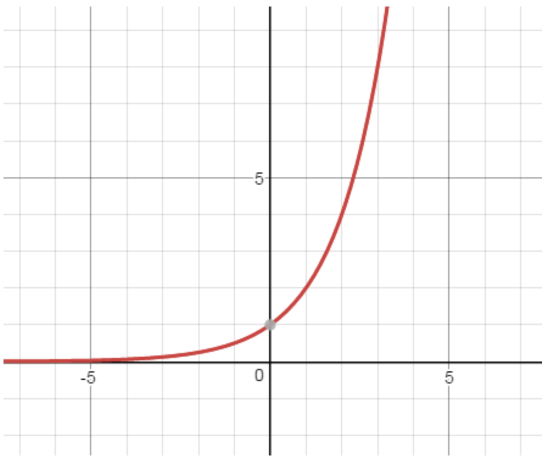
<u>Graphical representation:</u>

When $0 < a < 1$



- As x increase, $f(x)$ heads to 0
- As x increase, $f(x)$ heads to ∞
- Strictly decreasing
- Horizontal Asymptote along x-axis

When $a > 1$



- As x increase, $f(x)$ heads to ∞
- As x increase, $f(x)$ heads to 0
- Strictly increasing
- Horizontal Asymptote along x-axis

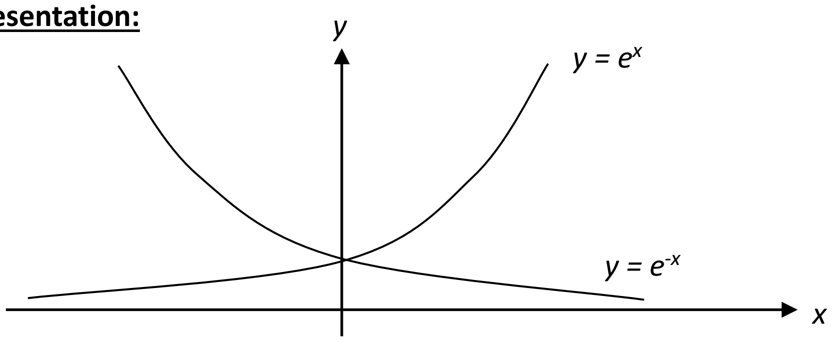
The natural exponential function is often used to model the quantity which grows or decays at a rate proportional to its value. It is very useful in modelling physical and chemical phenomenon as well as continuous compound interest. An irrational number e is defined as:

$$\begin{aligned} e &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \\ &= 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{r!}+\cdots \\ &= 2.718281828 \end{aligned}$$

The exponential function e^x is defined as:

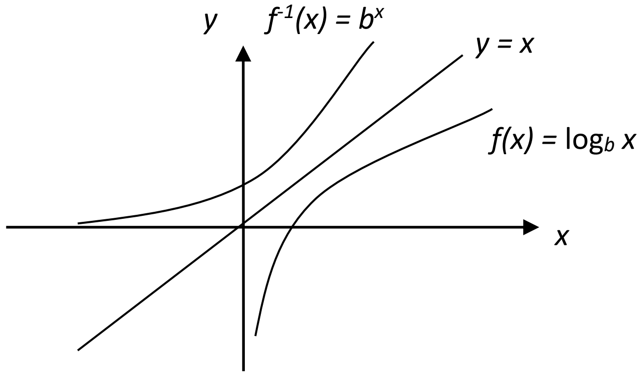
$$\begin{aligned} e^x &\equiv \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \\ &= 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^r}{r!}+\cdots \qquad \forall\, x \in \mathbb{R} \end{aligned}$$

Graphical representation:



2.3 Logarithms

Consider the functions $f(x)=\log_b x$ and $f^{-1}(x)=b^x$. The two functions are *inverse functions* of each other and they are symmetrical about the line $y=x$.



If b > 0, b ≠ 1 and y > 0, we change the subject of the formula $y=b^x \leftrightarrow log_b y = x$

Remark: $\boldsymbol{log_b y}$ is called the common logarithm of *y*. (i.e. $\boldsymbol{log_b\, y = log\, y}$)

Properties of logarithm

$$\log_b b = 1 \qquad \log_b 1 = 0$$
$$\log_b b^x = x \qquad b^{\log_b x} = x$$
$$\log_b \left(x^r\right) = r \log_b x$$
$$\log_b (xy) = \log_b x + \log_b y$$
$$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$$

The domain of $\log_b x$ is $x > 0$

Example:

Image how many of one number do we multiply to get another number?

$$5 \times 5 \times 5 = 125$$

Then

$$log_5 125 = log_5 5^3 = 3$$

The number we multiplying is called the *base*, the times we multiple is the logarithm. So, we can say the logarithm of 125 with base 5 is 3, or log base 5 of 125 is 3.

In such case, the logarithm tells us what the exponent is. In this example the base is 5 and the exponent is 3.

$$5^3 = 125$$

Example:

Find $log_8(0.125)$ and $log_5(0.008)$.

Notice $\frac{1}{8} = 0.125$, therefore $log_8(0.125) = -1$ is a negative logarithm that how many times to divide by the base number. Similarly $1 \div 5 \div 5 \div 5 = 5^{-3} = 0.008$, so $log_5(0.008) = -3$.

2.4 Common logarithms: Base 10

When the logarithm is written without a base, this is usually means that the base is 10 is many engineering context. It is called a common logarithm as on a calculator.

$$\log(1000) = log_{10}(1000) = 3$$

2.5 Natural logarithmic function

If the base $b = e$, then $\log_e y$ is called the *natural logarithmic function* and is denoted by $\ln y$. i.e.,

$$\log_e y = \ln y$$

Example:

Human population can be modelled over short period by unlimited exponential growth functions. If a country has a population of 22 million in 2000 and maintains a population growth rate of 1% per year, then its population in millions at a later time, taking t = 0 in 2000, can be modelled as $N(t) = 22e^{0.01t}$. Estimate the population in the year 2010.

In the year 2010, t = 10. Hence $N(10) = 22e^{0.01(10)} = 24.3$ the population is estimated to be 24.3 million.

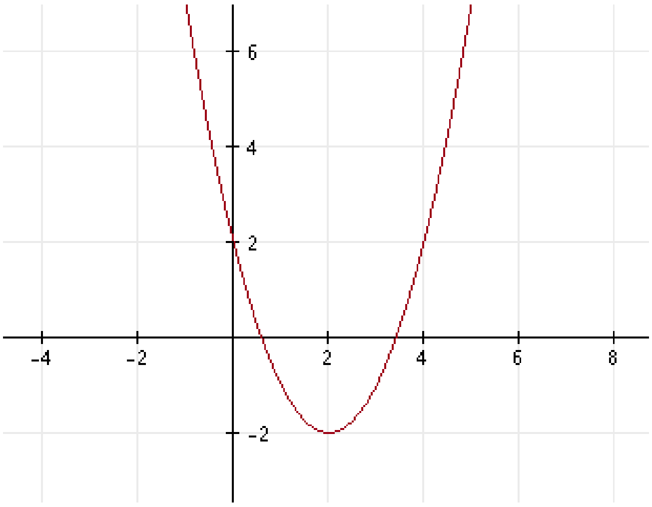
Practice:

1. Simplify $(\frac{e^x+e^{-x}}{2})^2 - (\frac{e^x-e^{-x}}{2})^2$
2. Simplify $\frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x-1) + \ln C$
3. Solve $5^{4-x} = 7^{3x+1}$ for x.
4. Solve $\frac{e^x-e^{-x}}{e^x+e^{-x}}$ = y for x in terms of y.
5. A radioactive isotope has a half-life of 35.2 years with function $\mathrm{Q(t)} = Q_0e^{-kt}$ at time t where Q_0 is the amount of substance at time 0. How many years (to the nearest tenth of a year) would it take before an initial quantity of 1 gram decay to 0.01 gram?

Topic 3 - Quadratic Equations and Polynomials

3.1 Solving Quadratic Equations by Algebraic Method

A quadratic function is any function specified by a rule that can be written *standard form* $\mathrm{a}x^2 + bx + c$, where *a* ≠ 0. Any quadratic function can be written in the form $\mathrm{a}(\mathrm{x}-\mathrm{h})^2 + k$ can be solved by completing square, and has a minimum value of k at point (h, k) for positive a.



(I) By Factor Method

Solve the quadratic equation $ax^2 + bx + c = 0$ where a ≠ 0 to form

$(\mathrm{m}x + \mathrm{n})(\mathrm{p}x + \mathrm{q}) = 0$ ➔ the roots are $x = -\frac{\mathrm{n}}{\mathrm{m}}$ or $x = -\frac{\mathrm{q}}{\mathrm{p}}$

(II) By Quadratic Formula

We can always find the solution from $ax^2 + bx + c = 0$ where a ≠ 0, using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the discriminant $(b^2 - 4ac)$ is

- Positive, then there are 2 real solutions
- Zero, there is 1 real solutions
- Negative, there are 2 complex solutions

(III) By Completing Square

Rewrite the quadratic equation $ax^2 + bx + c = 0$ where a ≠ 0 in the form $\mathrm{a}(x-\mathrm{h})^2 + k$,

$\mathrm{h} = \frac{\mathrm{b}}{2\mathrm{a}}$ and $\mathrm{k} = \mathrm{c} - \frac{\mathrm{b}^2}{4\mathrm{a}}$

➔ the roots are $x = \mathrm{h} \pm \sqrt{\frac{-k}{a}}$

Example:

Solve $5x^2 + 6x = -1$

Using **factor** method: $5x^2 + 6x + 1 = 0$

$\Rightarrow$ $(5x + 1)(x + 1) = 0$ then $\mathrm{m} = 5, \mathrm{n} = \mathrm{p} = \mathrm{q} = 1$

The roots are $x = -\frac{\mathrm{n}}{\mathrm{m}} = -\frac{1}{5} = -0.2$ or $x = -\frac{\mathrm{q}}{\mathrm{p}} = -\frac{1}{1} = -1$

Using **quadratic formula** method: $\mathrm{a} = 5, \mathrm{b} = 6, \mathrm{c} = 1$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{6^2 - 4(5)(1)}}{2(5)} = \frac{-6 \pm 4}{10}$$

Therefore $x = -0.2$ or $x = -1$

Using **completing square** method: $5x^2 + 6x + 1 = 0$, where $\mathrm{a} = 5, \mathrm{b} = 6, \mathrm{c} = 1$

$$\Rightarrow 5\left(x - \frac{6}{2(5)}\right)^2 + \left(1 - \frac{6^2}{4(5)}\right) = 0$$

So $x = \mathrm{h} \pm \sqrt{\frac{-k}{a}} = \frac{6}{2(5)} \pm \sqrt{\frac{-(1 - \frac{6^2}{4(5)})}{5}} = 0.6 \pm 0.4$ $\Rightarrow$ $x = -0.2$ or $x = -1$

Practice:

Solve the following quadratic equations

a) $6x^2 + 7x - 3 = 0$

b) $x(10x - 1) = 2$

c) $15x^2 - 26x = 21$

d) $x^2 + 20x + 29 = 0$

3.2 Solving Simultaneous Linear Equation in Two Unknowns

(I) By Substitution 代入法

Solve $\begin{cases} x + y = 5 & ---(1) \\ x - y = 1 & ---(2) \end{cases}$

From (1), $x = 5 - y \quad ---(3)$

Sub (3) into (2)

$$\begin{aligned}(5 - y) - y &= 1 \\ 5 - 2y &= 1 \\ y &= 2\end{aligned}$$

Sub y = 2 into (3), $x = 5 - 2 = 3$ $\quad \therefore \begin{cases} x = 3 \\ y = 2 \end{cases}$

(II) By Elimination 消元法

Solve $\begin{cases} x + y = 5 & ---(1) \\ x - y = 1 & ---(2) \end{cases}$

(1) + (2)

$$\begin{array}{rr} & x + y = 5 \\ +) & x - y = 1 \\ \hline & 2x = 6 \\ & x = 3 \end{array}$$

Sub x = 3 into (1), $y = 5 - 3 = 2$ $\quad \therefore \begin{cases} x = 3 \\ y = 2 \end{cases}$

3.3 Polynomials

A polynomial is an expression that can be written as a term or a sum of more than one term of the form $ax_1^{n_1}x_2^{n_2} \dots x_m^{n_m}$ where the *a* is a constant and the $x_1, \dots, x_m$ are variables. The degree of a term in a polynomial is the exponent of the variable, or, if more than one variable is the sum of the exponents of the variables. The degree of a polynomial with more than 1 term is the largest degrees of individual terms

Example:

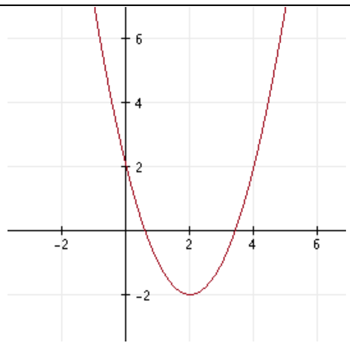
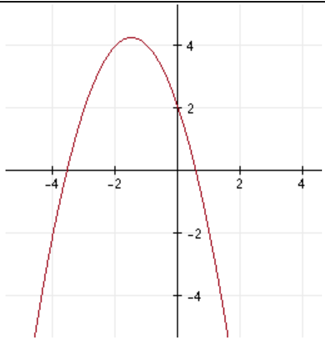
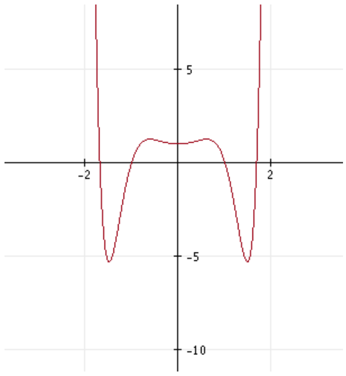
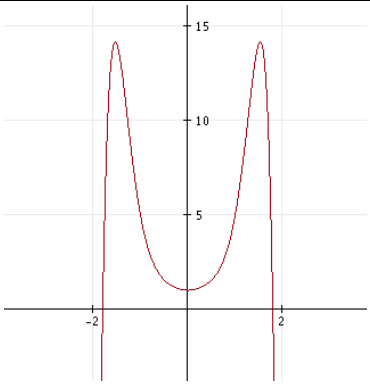
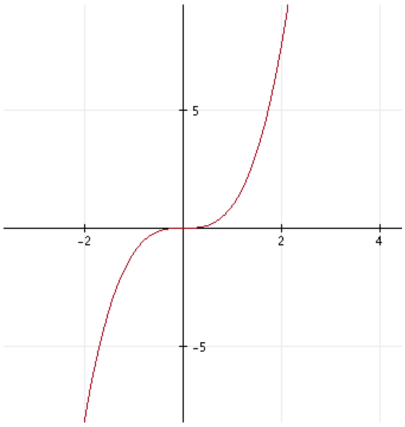
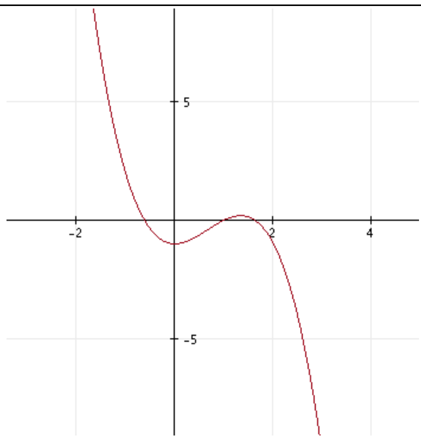
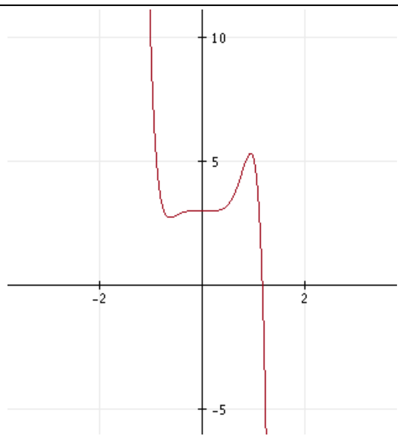
$2x^8$ has degree 8 $\qquad$ $10xy^2z^2 + 3x^4$ has degree 5 $\qquad$ π has degree 0

Practice:

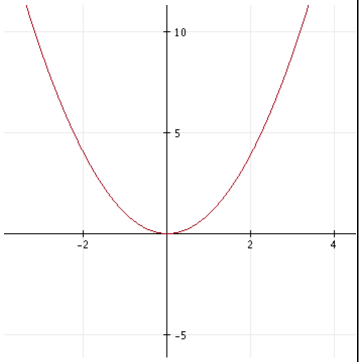
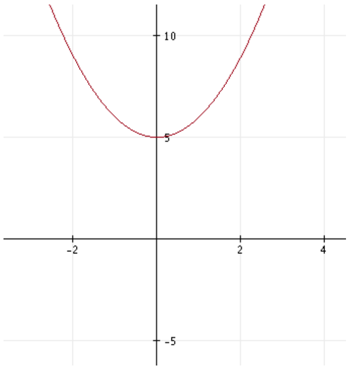
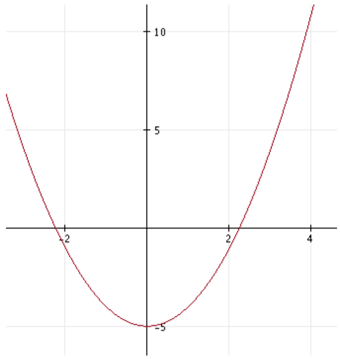
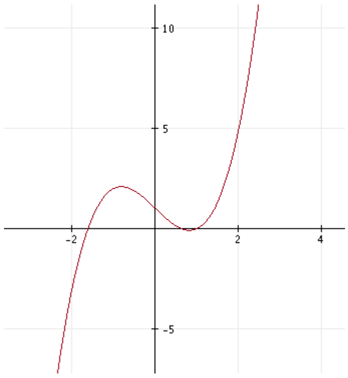
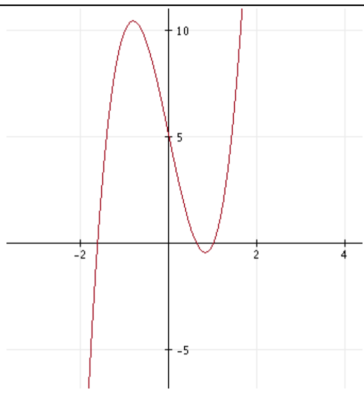
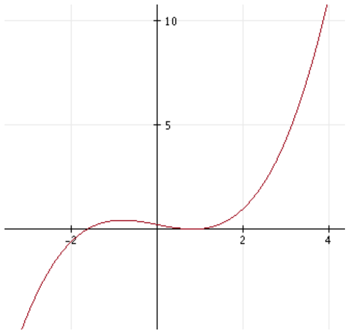
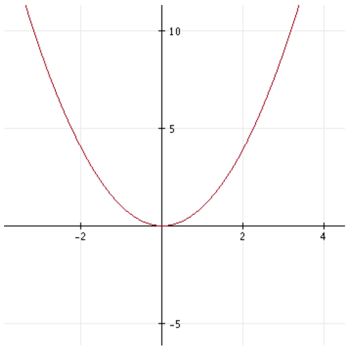
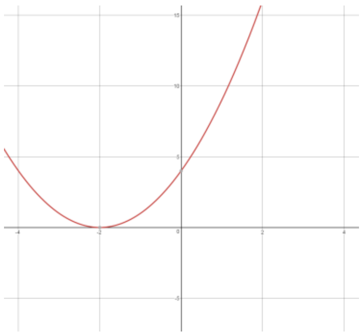
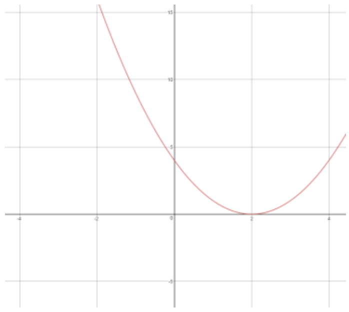
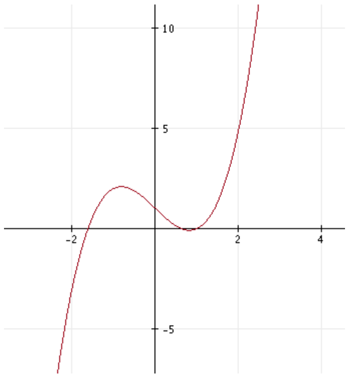
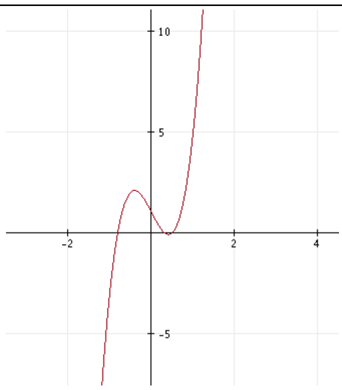
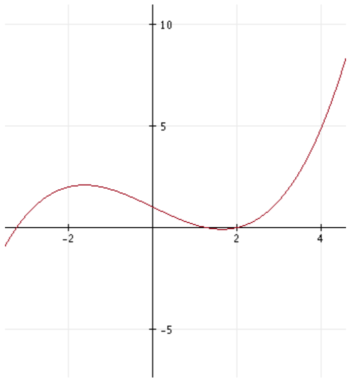
Simplify (a) $\frac{(8x^2y^{2/3})^{2/3}}{2(x^{3/4}y)^3}$ (b) $5x^3(3x+1)^{2/3}+3x^2(3x+1)^{5/3}$

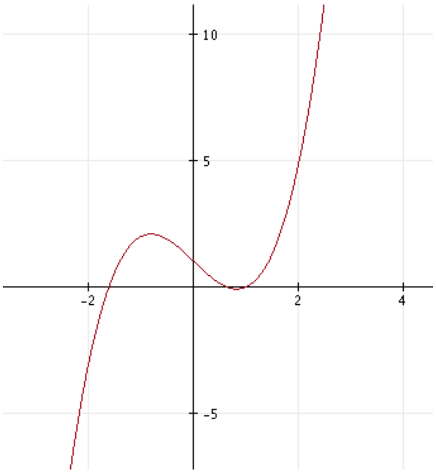
(c) $\frac{2}{x-1}+\frac{3}{x+1}-\frac{4x-2}{x^2-1}$ (d) $\frac{\frac{x}{x-1}-\frac{x}{x+1}}{\frac{x}{x-1}+\frac{x}{x+1}}$ (e) $\frac{3(x^2-9)^{\frac{1}{3}}-(4x)(\frac{1}{3})(x^2-9)^{-\frac{2}{3}}(2x)}{[(x^2-9)^{\frac{1}{3}}]^2}$

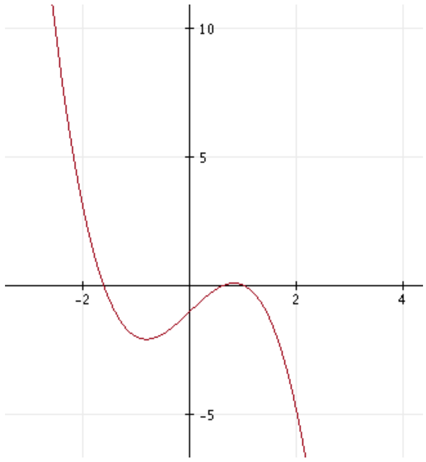
Behaviour of polynomials

	Positive leading coefficient	Negative leading coefficient
Even Degree polynomials	 $f(x) = x^2 – 4x +2$	 $f(x) = -x^2 – 3x +2$
	 $f(x) = x^8 - 3x^6 + x^2 + 1$	 $f(x) = -x^8 + 3x^6 + 2x^2 + 1$
Odd Degree polynomials	 $f(x) = x^3$	 $f(x) = -x^3 + 2x - 1$
	 $f(x) = 4x^7 - 5x^5 + 7x^2 + 5$	 $f(x) = -12x^7 + 4x^6 + 10x^5 + 3$

3.4 Transformation of functions

Vertical shift **f(x) + k**	 $f(x) = x^2$	 for $k > 0$ $f(x) = x^2 + 5$	 for $k < 0$ $f(x) = x^2 - 5$
Vertical stretch / compress **af(x)**	 $f(x) = x^3 – 2x + 1$	 for $a > 1$, $f(x) = 5(x^3 –2x +1)$	 for $\frac{1}{a}$, $a > 0$, $\frac{1}{5}f(x)$
Horizontal shift **f(x + k)**	 $f(x) = x^2$	 for $k > 0$ $f(x) = (x + 2)^2$	 for $k < 0$ $f(x) = (x - 2)^2$
Horizontal stretch / compress **f(ax)**	 $f(x) = x^3 – 2x + 1$	 for $a > 1$ $f(x) = (2x)^3 – 2(2x) + 1$	 for $a \in (0,1)$ $f(x) = (\frac{1}{2}x)^3 - 2(\frac{1}{2}x) + 1$

<table>
<tr>
<td>Reflection over x-axis</td>
<td>

$f(x) = x^3 – 2x + 1$</td>
<td>

$-f(x) = -(x^3 – 2x + 1) = -x^3 + 2x - 1$</td>
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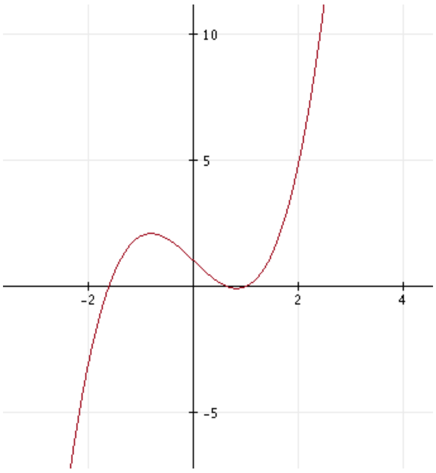
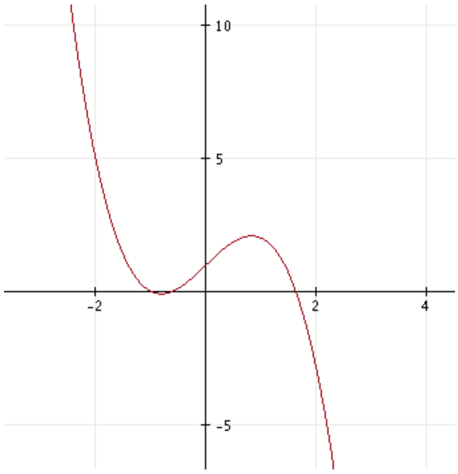
<tr>
<td>Reflection over y-axis</td>
<td>

$f(x) = x^3 – 2x + 1$</td>
<td>

$f(-x) = (-x)^3 – 2(-x) + 1 = -x^3 + 2x + 1$</td>
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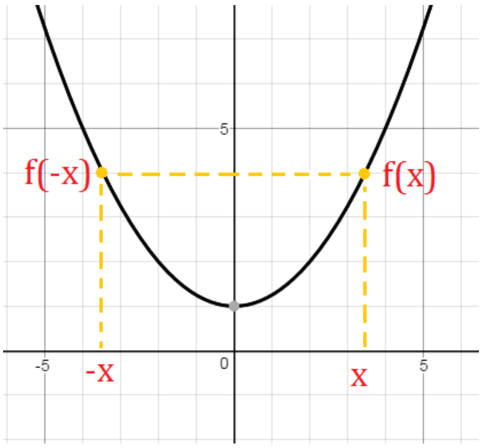
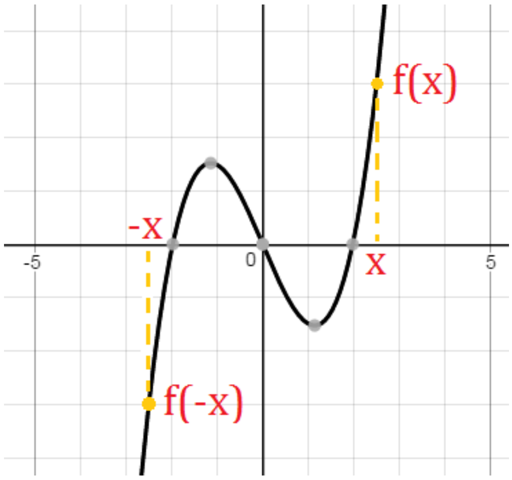
3.5 Odd and Even Function

Let f be a function, then

f is **even** if $f(-x) = f(x)\ \forall\ x$ in the domain of f



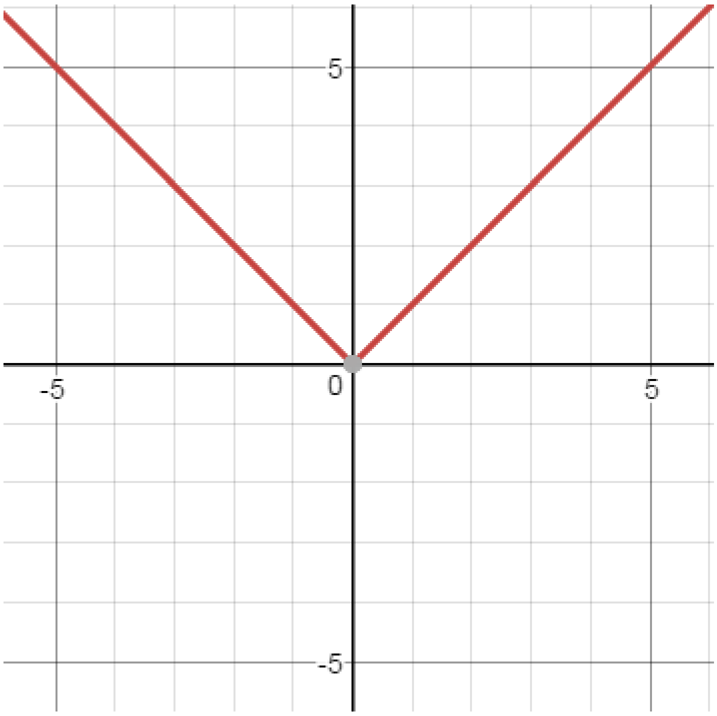
f is **odd** if $f(-x) = -f(x)\ \forall\ x$ in the domain of f



Practice:

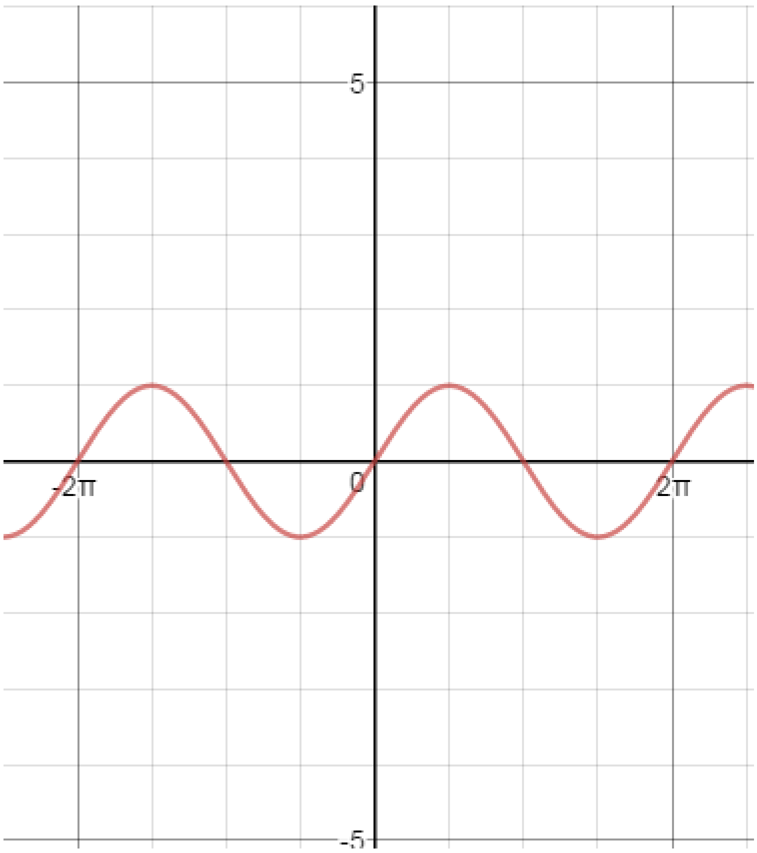
1. Given $y = f(x) = |x|$ is shown below. Sketch the graph of each function

a) $f(2x+1)$ b) $-0.5f(x)$



2. Given $y = f(x) = \sin x$ is shown below. Sketch the graph of each function

a) $f(0.5x+\pi)$ b) $-3f(x)-1$



Topic 4 - Differentiation

4.1 Limit

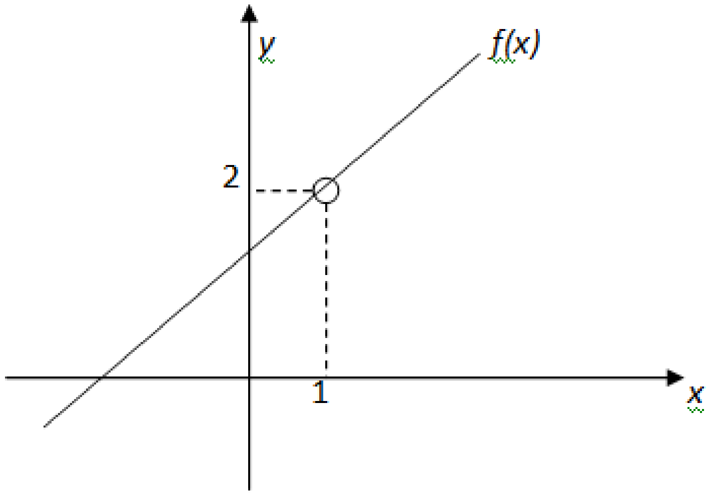
The idea of a limit

Given a function $f(x)$, we are often interested to know the behaviour of $f(x)$ when x approaches or tends to or is close to a fixed number c.

Remark: we are not talking about the value of f when x is equal to c.

Example:

What is the value of f when x is close to 1, where $f(x)=\dfrac{x^2-1}{x-1}$?



- When $x = 1$, $f(1)$ is undefined.
- But when x is close to 1, the value of f is close to 2.

Definition: Limit of a function

Left hand limit:

Given a function $f(x)$ and a fixed number c. If $f(x)$ approaches a fixed number M as x approaches (tends to) c from the left hand side of c, then the value M is called the *left hand limit* of $f(x)$.

$$\lim_{x\to c^-} f(x) = M$$

Right hand limit:

Given a function $f(x)$ and a fixed number c. If $f(x)$ approaches a fixed number N as x approaches (tends to) c from the right hand side of c, then the value N is called the *right hand limit* of $f(x)$.

$$\lim_{x\to c^+} f(x) = N$$

Existence of the limit:

> If the left hand limit of $f(x)$ is equal to the right hand limit of $f(x)$, say L, as x approaches c, then the limit of $f(x)$ exists and equals L. i.e.,
>
> If $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$, then $\lim_{x\to c} f(x) = L$

Remark: $\lim_{x\to c} f(x) = L$ means both left-hand and right-hand limits are equal to L.

$$\lim_{x\to c^-} f(x) = L \text{ and } \lim_{x\to c^+} f(x) = L$$

Limits approaching a value

Example:

Let $f(x) = x^2 - 1$, find the limit of $f(x)$ as $x \to 2$.

As $x \to 2^-$, $f(x) \to 3$ As $x \to 2^+$, $f(x) \to 3$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	2.61	2.9601	2.996001	3.004001	3.0401	3.41

$\because$ $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 3.$ Hence $\lim_{x\to 2} f(x) = 3$ (limit exists).

Example:

Let $f(x) = \dfrac{1}{x}$, find the limit of $f(x)$ as $x \to 0$.

As $x \to 0^-$, $f(x) \to -\infty$ As $x \to 0^+$, $f(x) \to +\infty$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-10	-100	-1000	1000	100	10

$\because$ $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x).$ Hence, $\lim_{x\to 0} f(x)$ does not exist.

Example:

Let $f(x) = \begin{cases} x+1 & \text{for } x \neq 1 \\ 3 & \text{for } x = 1 \end{cases}$, find $\lim_{x\to 1} f(x)$.

As $x \to 1^-$, $f(x) \to 2$ As $x \to 1^+$, $f(x) \to 2$

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2.001	2.01	2.1

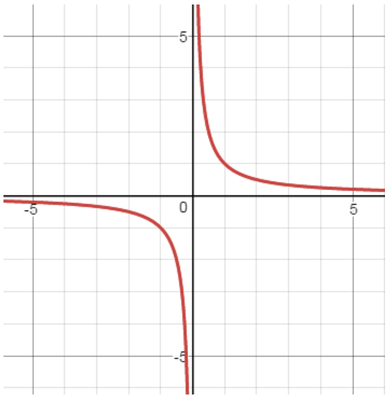
$\because$ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2.$ Hence, $\lim_{x\to 1} f(x) = 2$ (limit exists).

However, $\lim_{x\to 1} f(x) = 2 \neq f(1) = 3$

Limits approaching infinity

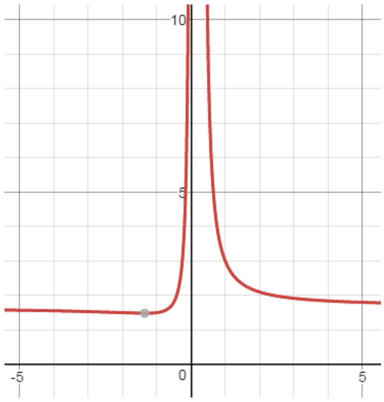
Example:

a) $\lim\limits_{x\to\infty} \frac{1}{x} = 0$



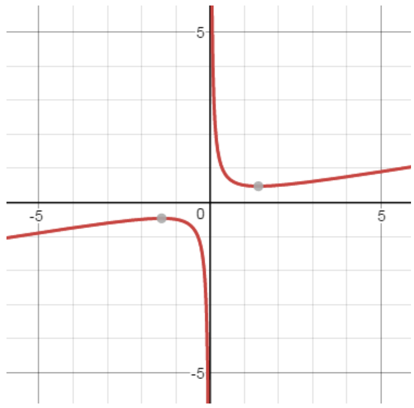
When the degree of the function is less than 0, the limit is 0

b) $\lim\limits_{x\to\infty} \frac{5x^2+1}{3x^2-x} = \frac{5}{3}$



In rational function, when the degree of the terms with the largest exponent is same, divide the coefficients.

c) $\lim\limits_{x\to\infty} \frac{x^3+2x}{6x^2}$



When the degree of the function is greater than 0, the limit is ∞ (or $-\infty$)

Practice:

Evaluate the following limits

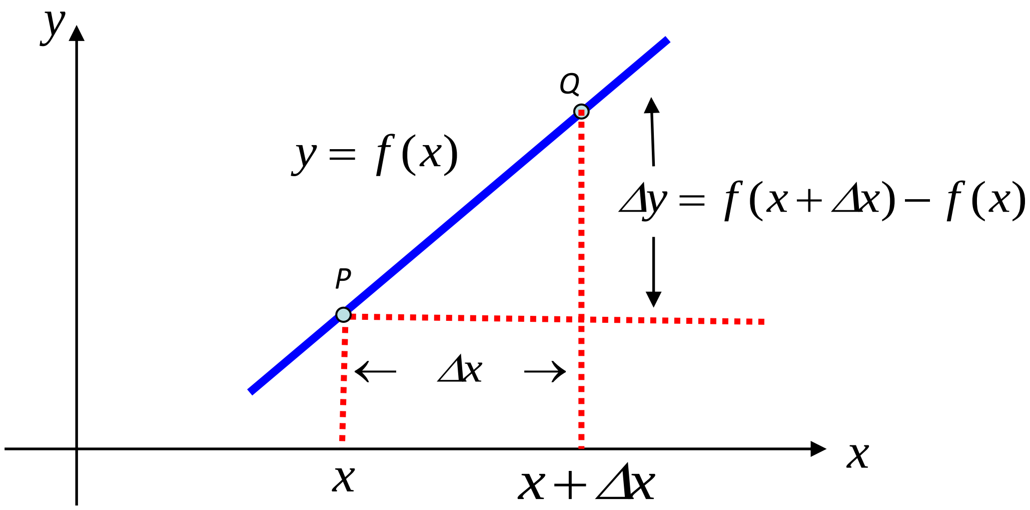
a) $\lim\limits_{x\to 2} \frac{x^2-4}{x-2}$ b) $\lim\limits_{x\to 1} \frac{x^2-1}{x^2+x-2}$ c) $\lim\limits_{x\to 1} f(x)$ when $f(x) = \begin{cases} x-2 & \text{for } x \le 1 \\ x^2 & \text{for } x > 1 \end{cases}$

d) $\lim\limits_{x\to\infty} \frac{8}{5-2x^3}$ e) $\lim\limits_{x\to\infty} \frac{x^2-4}{x-2}$ f) $\lim\limits_{x\to\infty} \frac{4x^3+x}{x^2-2x^3}$

4.2 Derivatives

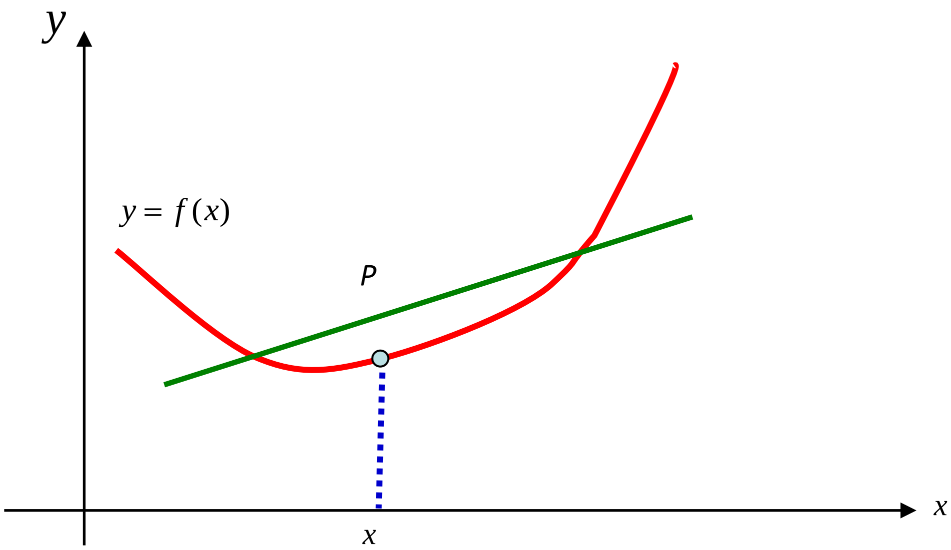
Concept: Consider the slope of a linear function at point *P*.

Let Δx denotes the small increment of x.



$$Slope\,of\;PQ = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

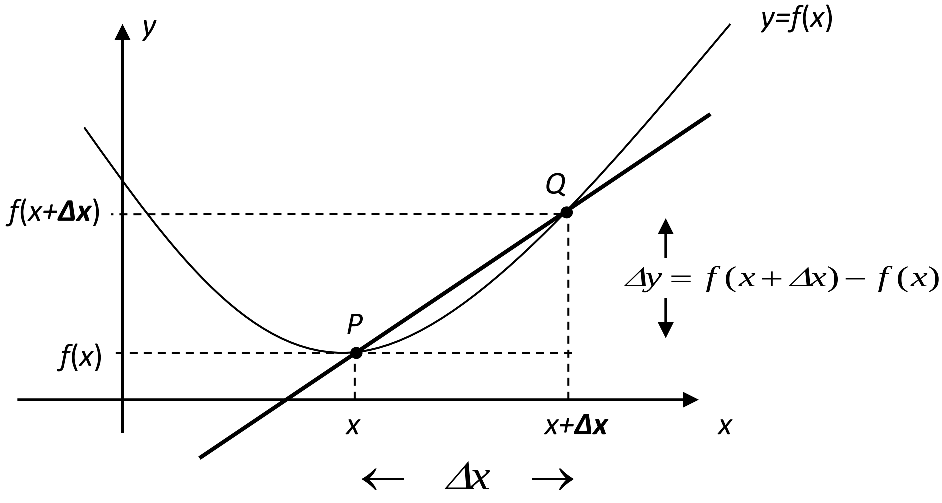
For a non-linear function $y = f(x)$, the slope of $f(x)$ generally changes at different points. How to describe the slope of $f(x)$ at point *P* ?



Important: The slope of the tangent line at point *P* is the slope of *f(x)* at point *P*.

Goal: Find the tangent line at point *P* and compute its slope.

Idea of derivatives: *how to find the slope of y = f(x) at point P ?*



To describe the slope at point P is equivalent to describe slope of the tangent line at point P.
As $\Delta x \to 0$, the straight line PQ is actually the tangent line at point P.

<u>Differentiation from the First Principles:</u>

> The derivative of a function $y = f(x)$ at the point x is given by
>
> $$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
>
> or
>
> $$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ for convenience.
>
> where $\frac{dy}{dx}$ is called the *first derivative* of y with respective to (w.r.t.) x.

Remarks:

1. We also use $\frac{dy}{dx}$, $\frac{df(x)}{dx}$, y' to denote the derivative of y w.r.t. x.
2. In one sense, derivative is the slope or gradient of the function $f(x)$.
3. For linear functions (i.e. straight line), the slope (or gradient) of the function at any point is the same.
4. For non-linear functions (e.g. quadratic equation), the slope (or gradient) of the function at any point <u>may not</u> be the same.
5. Derivative is the **<u>rate of change</u>** of one variable with respect to other variable.
6. Differentiation is a process of finding derivatives.

<u>**Successive Differentiation**</u>

Let $y = f(x)$ is a differentiable function of x, $\frac{dy}{dx}$ is the first derivative of *y* w.r.t. x. Then $\frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2}$ is called the <u>*second derivative*</u> of *y* w.r.t x. In general, we have the n th derivative of y w.r.t. x, denoted by $\frac{d^n y}{dx^n}$ ($f^{(n)}(x)$ or $y^{(n)}$). Sometimes, we will use y' or f' to denote the first derivative and y'' or f'' to denote the second derivative.

4.3 Fundamental Formulae for Derivatives

Note that $\frac{d}{dx}$ is a symbol and is called a differential operator. Based on the *First Principles,* we obtain the following formulae that can help you to find the derivatives directly.

1. $\frac{d}{dx}(C) = 0$ where C is a constant.
2. $\frac{d}{dx}(x) = 1$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$ where n can be any real number.

Let $f(x)$, $g(x)$ and $f(g(x))$ be differentiable functions of x.

4. $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$
5. $\frac{d}{dx}[C\,f(x)] = C\frac{d}{dx}[f(x)]$ where C is a constant.

---

6. $\frac{d}{dx}[f(x) \cdot g(x)] = [\frac{d}{dx}f(x)] \cdot g(x) + f(x) \cdot [\frac{d}{dx}g(x)]$
7. $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{[\frac{d}{dx}f(x)] \cdot g(x) - f(x) \cdot [\frac{d}{dx}g(x)]}{g(x)^2}$
8. $\frac{d}{dx}f(g(x)) = \frac{d}{dg(x)}f(g(x)) \times \frac{d}{dx}g(x)$

Example:

If $y = 3\sqrt{x} + 5x^3 - 2x + 10$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left[3x^{\frac{1}{2}} + 5x^3 - 2x + 10\right]$$
$$= \frac{d}{dx}(3x^{\frac{1}{2}}) + \frac{d}{dx}(5x^3) - \frac{d}{dx}(2x) + \frac{d}{dx}(10)$$
$$= 3\frac{d}{dx}(x^{\frac{1}{2}}) + 5\frac{d}{dx}(x^3) - 2\frac{d}{dx}(x) + \frac{d}{dx}(10)$$
$$= \frac{3}{2}(x^{-\frac{1}{2}}) + 15x^2 - 2(1) + 0$$
$$= \frac{3}{2}(x^{-\frac{1}{2}}) + 15x^2 - 2$$

Example:

If $f(t) = \frac{4t^3 - 3t + 2}{t}$, find $f'(t)$.

$$f'(t) = \frac{d}{dt}\left[\frac{4t^3 - 3t + 2}{t}\right]$$
$$= \frac{d}{dt}[4t^2 - 3 + 2t^{-1}]$$
$$= 4(2t) - 0 + 2(-1)t^{-2}$$
$$= 8t - 2t^{-2}$$

Practice:

Find the derivative $\frac{dy}{dx}$ of the following:

(a) $y = \frac{3x^2 - 2}{x^3}$ (b) $y = \frac{5}{\sqrt[4]{x}} + 2x^{-3} - x^{0.1}$

(c) $y = \frac{4\sqrt{x} - 3\sqrt[3]{x}}{6\sqrt[6]{x}}$ (d) $y = 3x^{\frac{1}{2}} - \frac{1}{5x^{\frac{1}{4}}} - \frac{\pi}{x^8}$

Topic 5 - Descriptive Statistics

5.1 What is Statistics?

Statistics is the branch of mathematics that transforms numbers into useful information for decision making. Statistics lets you know about the risks associated with making a business decision and allows you to understand and reduce the variation in the decision making process. Statistics involves collecting, classifying, summarizing, organizing, analyzing and interpreting numerical information.

Descriptive Statistics utilizes numerical and graphical methods to look for patterns in a data set, to summarize the information revealed in a data set, and to present information in a convenient form.

Inferential Statistics utilizes sample data to make estimates, decisions, predictions, or other generalizations about a larger set of data.

5.2 Some Terminologies in Statistics

- A *variable* is a characteristic of an item or individual.
- *Data* are the different values associated with a variable.
- A *population* consists of all the items or individuals about which you want to reach conclusion.
- A *sample* is the portion of a population selected for analysis.
- A *parameter* is a measure that describes a characteristic of a population
- A *statistic* is a measure that describes a characteristic of a sample.
- A *statistical inference* is an estimate or prediction or some other generalization about a population based on information contained in a sample.

5.3 Measure of Central Tendency

A. Ungrouped Data

To find the values from the *raw data (data are not grouped into categories)*

(I) Mean

Let $x_1, x_2, \cdots, x_N$ be N **population** elements.
Then,
the population mean $\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$;

Take a random **sample** from the population, say $x_1, x_2, \cdots, x_n$
Then,

the sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

(II) Mode

Mode is the data with the highest frequency obtained by observation. Note that a distribution may have more than one mode.

Example:

1, 1, 2, 2, 3, 3, 3, 4, 4, 4, 5 mode = 3 and 4

(III) Median

Median is the middle value in an ordered sequence of data.
Therefore, when we calculate the median, we need to arrange the data in *ascending* order first.

Example:

1.2 , 1.4 , 1.8 , 2.1 , 2.7 , 3.5 , 3.9
$n = 7$ (odd number of data), median = 2.1

Example:

1.2, 1.4, 1.8, 2.1, 2.7, 3.5, 3.9, 4.1
$n = 8$ (even number of data), median $= \frac{1}{2}(2.1 + 2.7) = 2.4$

In general,

For *n* data, the median $= \frac{(n+1)^{th}}{2}$ data

Remark: *Mean, Mode and Median* are the most common parameters to measure the *central tendency* of a set of data.

(IV) Lower quartile Q_1 and Upper quartile Q_3

$Q_1 = \frac{1}{4}(n+1)^{th}$ data and $Q_3 = \frac{3}{4}(n+1)^{th}$ data

Example:

10 25 30 30 35 40 45 70

$Q_1 = \frac{1}{4}(8+1)^{th} = 2.25^{th}$ data = 2nd data (round off) = 25

$Q_3 = \frac{3}{4}(8+1)^{th} = 6.75^{th}$ data = 7th data (round off) = 45

Hence $Q_2 = \frac{2}{4}(8+1)^{th} = 4.5^{th} = \frac{1}{2}(30+35) = 32.5$

In general, the ***p*^th^ *quartile*** is:

$$\boxed{Q = \frac{p}{100}(n+1)^{th} \text{ data}}$$

(V) Interquartile range (I.Q.R.)

Interquartile range is defines as $Q_3 - Q_1$. This will be discussed in the next section.

In the above example, I.Q.R $= Q_3 - Q_1 = 45 - 25 = 20$

Practice:

1. The readings of diastolic blood pressure (unit: mm Hg) of 16 randomly selected males aged 35 to 44 were

66	70	74	75	79	81	81	82
85	91	91	93	95	99	99	100

(a) Find the mean, mode, median, Q_1 and Q_3.

(b) Find also the 10th and 85th percentile of the readings.

2. The means and the number of observations obtained from three sets of data *Set A*, *Set B* and *Set C* are 15, 20 and 24 on the basis of 30, 35 and 50 observations respectively. If these three sets of data are combined, then what is the combined mean?

5.4 Measures of Dispersion

Common Measures of dispersion

(I) Range

Range is the difference between the largest and the smallest value.

The larger the range, the larger is the variation of the data.

Example:

72 , 16 , 23 , 6 , 55 , 45

Range = 72 – 6 = 66

(II) Interquartile Range (I.Q.R.)

It measures the variation of the middle 50 % of the data.

$$\text{I.Q.R.} = Q_3 - Q_1$$

where Q_1 and Q_3 are the first and the third quartiles respectively.

(III) Mean Absolute Deviation (MAD)

MAD is a measure of deviation from the mean.

$$\text{MAD} = \frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}|$$

(IV) Variance (σ^2 ***)** *and* **Standard Deviation (*** σ)

Variance is another measure of deviation from the mean. In order to avoid the deviations cancelling of each other, we have to square each difference.

For **population variance**: $$\sigma^2 = \frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N}$$

where N is the total number of data in the *population*

μ is the *population* mean

For **sample variance**: $$\sigma_{n-1}^2 \text{ or } s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

where n is the total number of data in the *sample*

$\bar{x}$ is the *sample* mean

Remarks:

(1) When the population mean μ and population variance σ^2 are unknown, sample mean $\bar{x}$ and sample variance s^2 are used as their estimators respectively.

(2) Divide the sum by n tends to underestimate σ^2

(3) Larger variance $\Rightarrow$ Larger variation

(4) Both s and σ have the same unit as the data set.

Standard deviation is the square root of the variance. It measures the average squared deviation of the observations from the mean.

Example:

Consider the set of data of a population:

66 70 74 75 79 81 81 82

Then,

$$\mu = \frac{66+70+\ldots+82}{8} = 76$$

$$\sigma^2 = \frac{(66-76)^2+(70-76)^2+\ldots+(82-76)^2}{8} = 29.5$$

$$\sigma = \sqrt{29.5} = 5.4314$$

Example:

Two data sets:

D1: {12 6 15 3 12 6 21 15 18 12}

D2: {12 10 12 14 10 13 12 11 14 12}

Then,

$$\bar{x}_1 = \frac{12+6+\cdots+12}{10} = 12$$

Range = $21-6=15$

MAD = $\frac{1}{10}\left(|12-12|+|6-12|+\cdots|12-12|\right) = 4.2$

$$s^2 = \frac{(12-12)^2+(6-12)^2+\ldots+(12-12)^2}{9} = 32$$

$$s = \sqrt{32} = 5.66$$

Similarly, for the second data set: $\bar{x}_2 = 12$, range = 4, MAD = 1, $s^2 = 2$ and $s = 1.414$. Therefore, the two data sets have the same sample mean but different deviation from the mean.

Note: μ, σ and s can be obtained directly by using calculator. The corresponding symbols on the calculator are summarized below:

		Symbol on Calculator
Population	Mean: μ	$\bar{x}$
	s.d: σ	${}_x\sigma_n$
Sample	Mean: $\bar{x}$	$\bar{x}$
	s.d: s	${}_x\sigma_{n-1}$

Remarks: Computational formula for s :

$$\sum_{i=1}^{n}(x_i-\bar{x})^2=\sum_{i=1}^{n}x_i^2-\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2$$

Practice:

1. The weights of a random sample of ten sodium hydrogen tablets are (in grams)

0.469	0.442	0.446	0.416	0.444
0.366	0.430	0.425	0.400	0.445

Find the mean, variance and standard deviation of the sample.

2. Suppose a data set with $n = 6$ is {17, 5, 4, 10, 2, 11}.
   (a) Compute the mean, median and standard deviation.
   (b) Now, the $\sum x_i, \sum x_i^2$ and the minimum of three additional data are 75, 1937 and 20 respectively. What are the sample mean, sample median and the sample standard deviation of the combined set of these 9 data?

Selected Answer to Practice Question

Topic 1:

1. $\mathbb{N}$: $\sqrt{625}$ $\mathbb{Z}$: $-5, 0, \sqrt{625}$ $\mathbb{Q}$: $-5, -\frac{5}{3}, 0, \frac{50}{7}, \sqrt{625}$

2. a) 1.99 b) 40314

3. a) 1/8 b) $\frac{1}{\frac{2}{5}+\frac{1}{2}} = \frac{10}{9}$

Topic 3:

1. 1

2. $\ln C\sqrt{\frac{x-1}{x+1}}$

3. $x = \frac{4\ln 5 - \ln 7}{3\ln 7 + \ln 5} \approx 0.60$

4. $x = \frac{1}{2}\ln\frac{1+y}{1-y}\ for\ -1 < y < 1$

5. $\mathrm{t} = \frac{-35.2\ln 0.01}{\ln 2} \approx 233.9\ years$

Topic 4.1:

a) x = 1/3 or -1.5

b) x = 0.5 or -0.4

c) $\mathrm{x} = \frac{7}{3}$ or $\mathrm{x} = -\frac{3}{5}$

d) x = -5 + 2i or x = -5 - 2i

Topic 4.3:

a) $\frac{2}{x^{11/12}y^{23/9}}$

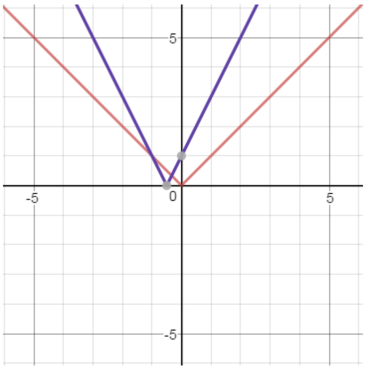
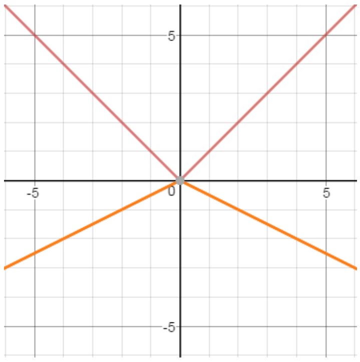
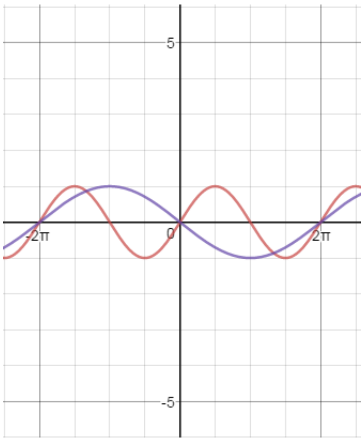
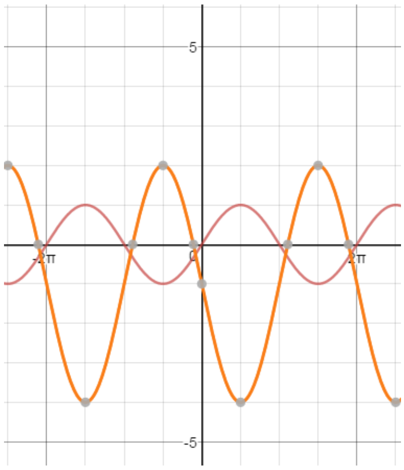
b) $x^2(3x+1)^{\frac{2}{3}}(14x+3)$

c) $\frac{1}{(x-1)}$

d) $\frac{1}{x}$

e) $\frac{(x^2-81)}{3(x^2-9)^{\frac{4}{3}}}$

Topics 4.5:

a) $f(2x+1)$ 	b) $-0.5f(x)$ 
2a) $f(0.5x+\pi)$ 	2b) $-3f(x)-1$ 

Topic 5.1:

a) 4

b) 2/3

c) Does not exist since -1 & 1

d) 0

e) Infinity

f) -2

Topic 5.3:

1. (a) Mean 85.0625; Mode 81,91 & 99; Median=8.5th=83.5 ; Q1=4.25th=75 ; Q3=12.75th=95
   (b) 10th percentile=1.7th=70 ; 85th percentile=14.45th= 99
2. 20.4348

Topic 6:

1. 0.428, 0.00084, 0.0289
2. mean = 26.3; Sd = 16.7812
3. (a) mean = 8.1667 median=3.5th = (5+10)/2=7.5 sd = 5.5648
   (b) a = 20, b = 24, c = 31